A note on the Hamiltonian constraint in canonical GR

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March 25, 2008

Abstract

The Hamiltonian constraint of the coupled Einstein–Yang–Mills–Higgs system with a cosmological constant is shown to be a pure Poisson bracket of a dimensionless functional on the phase space and the volume of the three-space. One of its potential consequences, a restriction on the eigenstates of the volume operator in a class of canonical quantum gravity theories, is also pointed out.

1 Introduction

It has been conjectured for a long time that the Chern–Simons functional, introduced originally in pure differential geometry [1], should play some fundamental role in physics (see e.g. [2]). In particular, the conformally invariant functional on Riemannian 3-manifolds [3] can be generalized for initial data sets of general relativity: the Chern–Simons functional built from an appropriate connection on the pull back to the Cauchy hypersurface of the spacetime tangent bundle is a global conformal invariant of the initial data set [4, 5]. The functional can also be introduced in the canonical formulation of general relativity too, where its conformal invariance implies the vanishing of its Poisson bracket with the 3-volume.

The Chern–Simons functional can be defined on the spinor bundle over the Cauchy hypersurface too. In this case it is complex, its real part is the previous functional, but its imaginary part is *not* a conformal invariant. However, this is connected with the vacuum general relativity: the Chern–Simons functional, defined in the spinor representation, is invariant with respect to infinitesimal conformal rescalings on every Cauchy surface precisely when the vacuum Einstein equations are satisfied [6]. This made it possible to reformulate the Hamiltonian constraint of vacuum GR: this constraint is the Poisson bracket of the spinor Chern–Simons functional and the 3-volume.

Recently Soo generalized the above result to include the cosmological constant, and he discussed its implications in a quantum version of canonical general relativity [7], too.

In the present paper we extend this result further by showing that the Hamiltonian constraint even for the coupled Einstein–Yang–Mills–Higgs system (with or without the cosmological constant) is the Poisson bracket of an appropriate dimensionless functional and the volume of the three-space. In a canonical quantum theory of gravity this yields a potential restriction on the volume eigenstates of the quantum volume operator.

Our conventions followed here are those of [6]. In particular, the three-metric is *negative* definite, the spacetime curvature is defined by $-{}^4R^a{}_{bcd}X^b := 2\nabla_{[c}\nabla_{d]}X^a$, when Einstein's equations take the form ${}^4G_{ab} = -\kappa T_{ab}$ with $\kappa := 8\pi G$, and the orientation of the induced volume 3-form of the spacelike hypersurface Σ with future pointing unit normal t^a is defined by $\varepsilon_{abc} := \varepsilon_{abcd}t^d$.

2 The Hamiltonian constraint

For the sake of simplicity the base manifold Σ is assumed to be closed. The canonical variables are $(q_{ab}, \tilde{p}^{ab})$, $(\phi^{\mathbf{i}}, \tilde{\pi}_{\mathbf{i}})$ and $(A^{\mathbf{i}}_{a\mathbf{j}}, \tilde{E}^{a\mathbf{i}}{}_{\mathbf{j}})$ in the gravitational, the Higgs and the Yang–Mills sectors, respectively. The Hamiltonian constraint of the coupled Einstein–Yang–Mills–Higgs system with the cosmological constant λ is

$$\tilde{\mathcal{C}} := \tilde{\mathcal{C}}_0 + \frac{\lambda}{\kappa}\sqrt{|q|} + \mu_H\sqrt{|q|} + \mu_{YM}\sqrt{|q|} = 0, \tag{2.1}$$

where $\tilde{\mathcal{C}}_0$ is the Hamiltonian constraint function of the vacuum GR and μ_H and μ_{YM} are the energy densities of the Higgs and Yang–Mills fields, respectively, given explicitly by

$$\tilde{\mathcal{C}}_0 := -\frac{1}{2\kappa}\Big(R - \frac{4\kappa^2}{|q|}\big(\tilde{p}^{ab}\tilde{p}^{cd}q_{ac}q_{bd} - \frac{1}{2}[\tilde{p}^{ab}q_{ab}]^2\big)\Big)\sqrt{|q|}, \tag{2.2}$$

$$\mu_H := \frac{1}{2}G^{\mathbf{ij}}\pi_{\mathbf{i}}\pi_{\mathbf{j}} - \frac{1}{2}G_{\mathbf{ij}}q^{ab}\big(\mathbb{D}_a\phi^{\mathbf{i}}\big)\big(\mathbb{D}_b\phi^{\mathbf{j}}\big) + U(\phi), \tag{2.3}$$

$$\mu_{YM} := -\frac{1}{2}q^{ab}\Big(E^{\mathbf{i}}_{a\mathbf{j}}E^{\mathbf{j}}_{b\mathbf{i}} + B^{\mathbf{i}}_{a\mathbf{j}}B^{\mathbf{j}}_{b\mathbf{i}}\Big). \tag{2.4}$$

Thus the gravitational canonical variables $(q_{ab}, \tilde{p}^{ab})$ are the ADM variables [8], R is the curvature scalar, and $\tilde{p}^{ab}$ is built from the metric and the extrinsic curvature χ_{ab} of Σ in the spacetime as $\tilde{p}^{ab} = -\frac{1}{2\kappa}(\chi^{ab} - \chi q^{ab})\sqrt{|q|}$. The Higgs field is a multiplet $\{\phi^{\mathbf{i}}\}$, $\mathbf{i} = 1, ..., n$, of real scalar fields on Σ, which are transformed among each other under the action of an n-dimensional representation of some compact gauge group. This representation of the gauge group is assumed to be a subgroup of $GL(n, \mathbb{R})$ leaving the symmetric positive definite metric $G_{\mathbf{ij}}$ fixed. The canonical momentum $\tilde{\pi}_{\mathbf{i}}$ is just $\sqrt{|q|}$-times $\pi_{\mathbf{i}}$, where the latter is

given in terms of the Lagrange variables $(\phi^{\mathbf{i}}, \dot\phi^{\mathbf{i}})$ and the lapse N and the shift N^e by $\pi_{\mathbf{i}} = -\frac{1}{N}G_{\mathbf{ij}}(\dot\phi^{\mathbf{j}} - N^e\mathbb{D}_e\phi^{\mathbf{j}})$. $\mathbb{D}_e$ is the gauge covariant derivative operator: $\mathbb{D}_e\phi^{\mathbf{i}} := D_e\phi^{\mathbf{i}} + A^{\mathbf{i}}_{e\mathbf{j}}\phi^{\mathbf{j}}$, and $U(\phi)$ is some potential, e.g. typically of the form $\frac{1}{2}m^2G_{\mathbf{ij}}\phi^{\mathbf{i}}\phi^{\mathbf{j}} + \frac{1}{4}\nu(G_{\mathbf{ij}}\phi^{\mathbf{i}}\phi^{\mathbf{j}})^2$ with the so-called rest mass m and self-interaction parameter ν. Finally, the canonical momentum $\tilde E^{a\mathbf{i}}{}_{\mathbf{j}}$ for the Yang–Mills field is $\sqrt{|q|}$-times the electric field strength $E^{a\mathbf{i}}{}_{\mathbf{j}}$, and $B^{a\mathbf{i}}{}_{\mathbf{j}}$ is the magnetic field strength. The other constraints (namely the momentum constraint of GR and the Gauss constraint of the Yang–Mills theory) will not play any role in the present paper.

On the gravitational sector of the phase space we introduce the real valued functions

$$V\big[N\big] := \int_\Sigma N\sqrt{|q|}\mathrm{d}^3x, \qquad T\big[f\big] := \frac{2}{3}\int_\Sigma f\tilde p^{ab}q_{ab}\mathrm{d}^3x \tag{2.5}$$

for any real, integrable functions N and f on Σ. If $D \subset \Sigma$ is any measurable set and N is its characteristic function, then $V[N]$ is the 3-volume of D. Essentially $V[N]$ is Misner's time function (more precisely, it is $-\frac{1}{3}\ln V[1]$), while $T[f]$ is the smeared version of York's time function. Their Poisson bracket is $\{T[f], V[N]\} = V[fN]$.

Let us fix a spinor structure on $T\Sigma$. Then the intrinsic Levi-Civita connection D_e and the extrinsic curvature χ_{ab} determine a connection $\mathcal{D}_e$ on the spinor bundle, the so-called Sen connection, according to $\mathcal{D}_e\lambda^A := D_e\lambda^A - \frac{1}{\sqrt{2}}\chi_e{}^A{}_B\lambda^B$, where the second index of the extrinsic curvature χ_{ef} has been converted to a (symmetric) pair AB of unitary spinor indices. Denoting the corresponding connection 1-form and curvature 2-form in some normalized dual spin frame $\{\varepsilon^A_{\underline{A}}, \varepsilon^{\underline{A}}_A\}$, $\underline{A} = 0, 1$, by $\Gamma^{\underline{A}}_{e\underline{B}}$ and $F^{\underline{A}}{}_{\underline{B}\,cd}$, respectively, the Chern–Simons functional is defined by

$$Y\big[\Gamma^{\underline{A}}{}_{\underline{B}}\big] := \int_\Sigma \Big(F^{\underline{A}}{}_{\underline{B}\,de}\Gamma^{\underline{B}}_{f\underline{A}} + \frac{2}{3}\Gamma^{\underline{A}}_{d\underline{B}}\Gamma^{\underline{B}}_{e\underline{C}}\Gamma^{\underline{C}}_{f\underline{A}}\Big)\frac{1}{3!}\delta^{def}_{abc}. \tag{2.6}$$

For its basic properties (in particular the change under the transformation of the spinor basis $\{\varepsilon^A_{\underline{A}}, \varepsilon^{\underline{A}}_A\}$ or the conformal rescaling of the spacetime metric, the calculation of its functional derivatives with respect to the canonical variables as well as a more detailed discussion of the geometric background) see [6]. What we need here is the result that Y modulo $8\pi^2$ is invariant with respect to the change of the spinor basis (and hence $\mathrm{Re}\,Y$ modulo $8\pi^2$ and $\mathrm{Im}\,Y$ are well defined real valued functions on the ADM phase space), and that

$$\Big\{\mathrm{Re}\,Y, V\big[N\big]\Big\} = 0, \qquad \Big\{\mathrm{Im}\,Y, V\big[N\big]\Big\} = \kappa^2\int_\Sigma \tilde{\mathcal{C}}_0N\mathrm{d}^3x; \tag{2.7}$$

i.e. the Hamiltonian constraint of the vacuum Einstein theory is the pure Poisson bracket of the Chern–Simons functional built from the Sen connection on the spinor bundle and Misner's time [6].

This result can be extended to Einstein–Yang–Mills–Higgs systems. In fact, if we define

$$G := \operatorname{Im} Y + T\bigl[\kappa\lambda\bigr] + \kappa^2 \frac{2}{3}\int_\Sigma \bigl(\mu_H + \mu_{YM}\bigr)\tilde{p}^{ab} q_{ab} \mathrm{d}^3x, \tag{2.8}$$

then by the Poisson bracket of the Misner and York times, equation (2.7) and the definitions it follows that

$$\Bigl\{G, V\bigl[N\bigr]\Bigr\} = \kappa^2 \int_\Sigma \tilde{C} N \mathrm{d}^3x =: \kappa^2\, H\bigl[N\bigr], \tag{2.9}$$

which is a generalization of the previous results of [6] for the vacuum, and of Soo [7] for the cosmological constant cases. Thus the geometric content of the Hamiltonian constraint is that G must be constant along the flow of the Hamiltonian vector field of $V[N]$. G, being dimensionless and depending on no smearing function, appears to be the 'universal generator function' by means of which the constraint governing the time evolution of the Einstein–Yang–Mills–Higgs system is generated. The lapse function N enters the dynamics only through $V[N]$.

Neither G nor $V[N]$ has weakly vanishing Poisson bracket with the constraints, and, in particular, with the Hamiltonian constraint. Thus they are *not* classical observables on the *whole* phase space. However, these Poisson brackets could be zero on certain *subsets* U of the constraint surface (e.g. at the points representing Einstein's static universe with a cosmological constant), and in this they behave as well defined classical observables on the special states represented by the points of U.

3 A restriction on the eigenstates of the quantum volume operator

The result (2.9) can be reformulated in Ashtekar's phase space, the starting point of most of the recent approaches of canonical quantum gravity [9]. Indeed, for the vacuum and the vacuum with cosmological constant cases this is already given in [6] and [7], respectively. (In fact, Soo used the even more general Barbero–Immirzi variables as the basic canonical coordinates.) Thus what remains be done is to express the metric q^{ab} in μ_H and μ_{YM} and the coefficient $\tilde{p}^{ab}q_{ab}$ of μ_H and μ_{YM} in the generator function G in terms of the Ashtekar variables, which is a straightforward calculation.

In the present section we intend to point out a simple consequence of (2.9) in a class of canonical quantum theories of general relativity that are based on Dirac's quantization of constrained systems. First, it is known that well defined quantum operators for both the area of a surface and the volume of a compact domain D in the three-space Σ can be introduced, and the spin network states are eigenstates of them [10, 11]. As we mentioned in connection with (2.5), any domain D can be characterized by its own characteristic function N, and the corresponding volume operator, acting as a linear operator on some complex representation space $\mathcal{V}$, will be represented in the same way and will be denoted

by $\widehat{V[N]}$. Next we will have three assumptions: 1. We assume that the classical generator functional G and the Hamiltonian constraint function $H[N]$ have a well defined operator form, $\widehat{G}$ and $\widehat{H[N]}$, respectively, acting on $\mathcal{V}$. (In fact, what we use in the subsequent discussion is that they are well defined on the volume eigenstates.) 2. Suppose that the classical Poisson bracket relation (2.9) still holds in operator form: $[\widehat{G},\widehat{V[N]}]=\mathrm{i}\hbar\,\kappa^2\,\widehat{H[N]}$. (Indeed, as Soo already showed [7], in the connection representation for an appropriate (symmetric) ordering there is an operator form $\widehat{H[N]}$ of the Hamiltonian constraint of the vacuum GR which is the commutator of the Chern–Simons operator and the three-volume. Hence *in this representation* our first two assumptions are satisfied.) 3. Following Dirac, we consider a state $|\Psi\rangle$ to be a *physical state* if and only if it is annihilated by the operator form of all constraints (and the space of these states will be denoted by $\mathcal{V}_0$), i.e. in particular $[\widehat{G},\widehat{V[N]}]\,|\Psi\rangle=0$ must hold for any physical state $|\Psi\rangle$.

Now let us consider an eigenstate $|\Psi_v\rangle$ of the volume operator $\widehat{V[N]}$ with eigenvalue v. Then the action of the operator form of (2.9) on the eigenstate $|\Psi_v\rangle$ gives

$$\widehat{V[N]}\Bigl(\widehat{G}|\Psi_v\rangle\Bigr)=v\Bigl(\widehat{G}|\Psi_v\rangle\Bigr)-\mathrm{i}\hbar\,\kappa^2\,\widehat{H[N]}|\Psi_v\rangle. \tag{3.1}$$

Since in general $\widehat{V[N]}$ is not (weakly) commuting with the constraint operators, the volume eigenstates are not expected to be physical states, and hence the second term on the right hand side of (3.1) is not vanishing. (The Hamiltonian constraint is not required to annihilate the volume eigenstates even if the volume is expected to be a quantum observable in the sense of Kuchař [12].) However, on certain subspaces $\mathcal{U}\subset\mathcal{V}_0$ the volume operator could be commuting with the constrains, and hence certain volume eigenstates could be physical states as well.

Thus suppose that $|\Psi_v\rangle$ is a physical state too, and hence it is annihilated by the Hamiltonian constraint operator. Therefore, by (3.1) *the volume eigenstate $|\Psi_v\rangle$ can be a physical state only if the operator $\widehat{G}$ maps the eigenstate $|\Psi_v\rangle$ into another eigenstate with the same eigenvalue v.* (To have a necessary and sufficient condition the other constraints also would have to annihilate $|\Psi_v\rangle$.) However, in general, without additional restrictions on $\widehat{G}$, the operators $\widehat{G}$ and $\widehat{V[N]}$ are *not* necessarily simultaneously diagonalizable even on the subspace spanned by the eigenstates $|\Psi_v\rangle$ with fixed v; and even if, in addition, every state in this subspace were a physical state: $\widehat{G}$ may still have a non-trivial Jordan form there. This restriction on the structure of the operator $\widehat{G}$ in these special states may help finding the operator form of the universal generator function G (and, in particular, of the Chern–Simons functional) in other (e.g. the loop) representations.

The author is grateful to Chopin Soo, Kornél Szlachányi and Péter Vecsernyés for the very useful, enlightening discussions on the quantum theoretic issues; and to the referees for their helpful criticism. This work was partially

supported by the Hungarian Scientific Research Fund (OTKA) grants T042531 and K67790.

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